

Causality and Characteristic Impedance

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Abstract- A new causal power-normalized waveguide equivalent-circuit theory fixes both the magnitude and phase of the characteristic impedance of a waveguide.

INTRODUCTION

We study the implications of the causal equivalent-circuit theory of [1] on the definition of characteristic impedance. That theory marries the power normalization of [2] with additional constraints [1] that enforce simultaneity of the theory's voltages and currents and the actual fields in the circuit. These additional constraints guarantee that the network parameters of passive devices in this theory are causal.

The additional constraints of the causal theory fix all of the free parameters in conventional waveguide circuit theories, including the characteristic impedance Z_0 , within a constant factor that defines the overall impedance normalization.

We will first summarize some of the essential results of [1]. We will then examine the implications of the theory on the definition of characteristic impedance, one of the important free parameters of conventional circuit theories fixed by causality.

In particular, we will show that in a lossless coaxial waveguide, Z_0 is real and constant. In lossless rectangular waveguide, Z_0 must be proportional to the wave impedance: the choice $|Z_0| = 1$ is not allowed. We will also investigate metal-insulator-silicon (MIS) transmission lines.

THE CAUSAL CIRCUIT THEORY

The power-normalized waveguide equivalent-circuit theory of [2] begins with a waveguide that is uniform in the axial direction and supports only a single mode of propagation at the reference plane where v and i are defined. The voltage v is defined by

$$\mathbf{E}_t(\mathbf{r}, z) = [c_+(z)e^{-\gamma z} + c_-(z)e^{\gamma z}] \mathbf{e}_t(\mathbf{r}) \equiv \frac{v(z)}{v_0} \mathbf{e}_t(\mathbf{r}) \quad (1)$$

and the current i by

$$\mathbf{H}_t(\mathbf{r}, z) = [c_+(z)e^{-\gamma z} - c_-(z)e^{\gamma z}] \mathbf{h}_t(\mathbf{r}) \equiv \frac{i(z)}{i_0} \mathbf{h}_t(\mathbf{r}), \quad (2)$$

where $\mathbf{r} = (x, y)$ is the transverse coordinate, \mathbf{E}_t and \mathbf{H}_t are the total electric and magnetic fields in the guide, \mathbf{e}_t and \mathbf{h}_t are the modal electric and magnetic fields of the single propagating mode, γ is the modal propagation constant, and c_+ and c_- are the forward and reverse amplitudes of the mode. The time dependence $e^{j\omega t}$ in (1) and (2) has been suppressed, and all of the parameters are functions of ω . The two factors v_0 and i_0 define v and i in terms of the fields and can be thought of as voltage and current normalization factors.

The total time-averaged power p in the waveguide is found by integrating the Poynting vector over the guide's cross section S :

$$p \equiv \frac{1}{2} \int_S \mathbf{E}_t \times \mathbf{H}_t^* \cdot \mathbf{z} dS = \frac{1}{2} \frac{v i^*}{v_0 i_0^*} \int_S \mathbf{e}_t \times \mathbf{h}_t^* \cdot \mathbf{z} dS. \quad (3)$$

The power normalization in [2] is achieved by imposing the constraint

$$v_0 i_0^* = p_0 \equiv \int_S \mathbf{e}_t \times \mathbf{h}_t^* \cdot \mathbf{z} dS, \quad (4)$$

which ensures that the time-averaged power is $p = \frac{1}{2} v i^*$.

The characteristic impedance Z_0 of a waveguide is defined by

$$Z_0 \equiv \left. \frac{v}{i} \right|_{c=0}. \quad (5)$$

Equations (1), (2), and (4) give

$$Z_0 = \frac{v_0}{i_0} = \frac{|v_0|^2}{p_0^*} = \frac{p_0}{|i_0|^2}. \quad (6)$$

Equation (6) shows that the phase of Z_0 in the power-normalized circuit theory is equal to the phase of p_0 , which is a fixed property of the guide uniquely determined by the modal field solutions \mathbf{e}_t and \mathbf{h}_t .

The causal circuit theory of [1] requires that $Z_0(\omega)$ be causal. That is, the theory requires that $\hat{Z}_0(t) = 0$ for $t < 0$, where $\hat{Z}_0(t)$ is the inverse Fourier transform of $Z_0(\omega)$. This condition ensures that the voltage in the waveguide responds to input currents after, not before, the onset of the current.

The theory also requires that $Y_0 \equiv 1/Z_0(\omega)$ be causal so that the current in the waveguide responds to input voltages after, not before, the onset of the voltage. These two constraints imply that $Z_0(\omega)$ is minimum phase [3], [4], [5].

The minimum phase constraint is a strong one. The real and imaginary parts of the complex logarithm of a minimum phase function are a Hilbert transform pair: that is, $\ln|Z_0|$ and $\arg(Z_0)$ are a Hilbert transform pair. The result is that we can determine $\ln|\lambda Z_0|$, where λ is

a constant, from the inverse Hilbert transform of $\arg(Z_0)$ [1].

The scalar multiplier λ sets the overall impedance of the system, is linked to the units of voltage and current chosen in the theory, and is the only free parameter not determined by the causal theory of [1].

LOSSLESS COAXIAL TRANSMISSION LINE

The power flow p_0 is real in a lossless coaxial transmission line, so the phase of Z_0 is 0. The set of constant functions form the null space of the Hilbert transform, so in the causal circuit theory $|Z_0|$ must be constant.

DOMINANT TE₁₀ MODE OF LOSSLESS RECTANGULAR WAVEGUIDE

The power flow p_0 and therefore Z_0 are real in a lossless rectangular waveguide above cutoff and imaginary below cutoff. So $\arg(Z_0)$ is equal to $\pm\pi/2$ below cutoff, and 0 above. The inverse Hilbert transform of a function that is equal to $-\pi/2$ for $-\omega_c < \omega < 0$, $\pi/2$ for $0 < \omega < \omega_c$, and 0 elsewhere is [6]

$$\frac{1}{2} \ln \left| \frac{\omega^2}{\omega^2 - \omega_c^2} \right|. \quad (7)$$

The causality constraint therefore requires that

$$|Z_0| \propto \sqrt{\left| \frac{\omega^2}{\omega^2 - \omega_c^2} \right|}, \quad (8)$$

where \propto indicates proportionality. That is, Z_0 must be proportional to the wave impedance of the guide: the choice $|Z_0| = 1$ is not admissible in the causal theory.

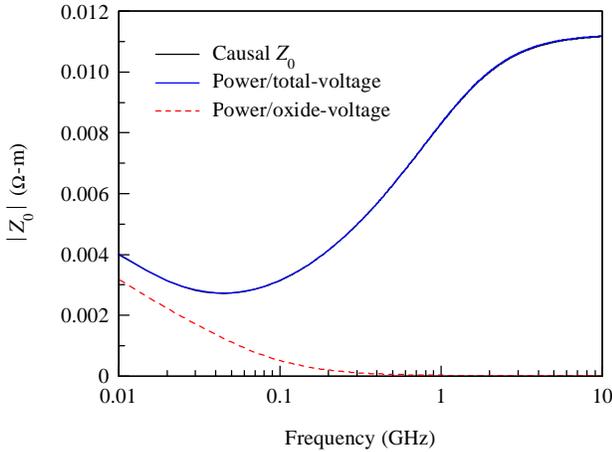


Fig. 1. $|Z_0|$ for the metal-insulator-semiconductor transmission line of [7]. The two solid curves are so close as to be indistinguishable.

MIS TRANSMISSION LINE

Different choices of voltage and current paths in conventional waveguide circuit theories result in different characteristic impedances in metal-insulator-semiconductor (MIS) transmission lines. Not all of these choices are consistent with causality.

Figure 1 compares three characteristic impedances for the TM_{01} mode of the infinitely wide MIS line investigated in [7]. This MIS line consists of a $1.0 \mu\text{m}$ thick metal signal plane with a conductivity of $3 \times 10^7 \text{ S/m}$ separated from the $100 \mu\text{m}$ thick $100 \Omega\text{-cm}$ silicon supporting substrate by a $1.0 \mu\text{m}$ thick oxide with conductivity of 10^{-3} S/m . The ground conductor on the back of the silicon substrate is infinitely thin and perfectly conducting.

The two solid curves in Fig. 1, which are labeled “Causal Z_0 ” and “Power/total-voltage,” agree so closely as to be indistinguishable on the graph. The curve “Causal Z_0 ” is the magnitude of the characteristic impedance determined from the phase of p_0 and the minimum phase properties of Z_0 . The curve “Power/total-voltage” is the magnitude of the characteristic impedance defined with a power-voltage definition. Here the power normalization is based on (4)

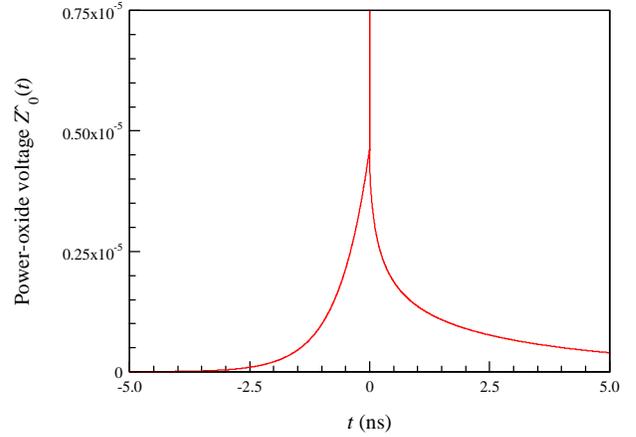


Fig. 4. The Fourier transform of the characteristic impedance labeled “Power/oxide-voltage” of Fig. 1.

(the integral of the Poynting vector over the guide cross section) and the voltage normalization on

$$v_0 = - \int_{\text{path}} \mathbf{e}_t \cdot d\mathbf{l}, \quad (9)$$

where the path begins at the ground on the back of the silicon substrate and terminates on the conductor metal on top of the oxide.

Conventional circuit theories do not specify voltage path uniquely, and the choice is not always obvious. For example, devices embedded in MIS lines are fabricated on the silicon surface; they are connected to the signal line with vias through the oxide and to the ground with ohmic contacts at the silicon surface. This suggests that a voltage path in the MIS line from the silicon surface through the oxide to the signal line, which is equally consistent with the conventional theories, might correspond more closely to the actual voltage seen by the device than the total voltage across the MIS line.

However, Fig. 1 shows that the characteristic impedance defined from the power constraint of (4) and the voltage across the oxide, which is labeled “Power/oxide-voltage,” differs significantly from the characteristic impedance required by the causal theory presented here. Figure 2 shows the Fourier transform of the characteristic impedance defined with the voltage path through the oxide and illustrates the difficulty with

this definition: the guide will respond to input signals before the excitation reaches it.

CONCLUSION

We have studied some of the implications of the causal power-normalized waveguide circuit theory of [1]. The examples illustrate an important contribution of the causal theory: it replaces the subjective and sometimes misleading “common-sense” criteria for defining Z_0 with a clear and unambiguous procedure that guarantees causal responses. This new approach should be especially useful in complex transmission structures where the choice of voltage and current paths are not intuitively obvious.

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